Series 10-Solutions

Exercise 1

The original system has two poles, one at $-1/\tau_1$ and the other at $1/\tau_2$. As such, the system is unstable. We would like to see how we can stabilize the system using different types of controllers.

The addition of a proportional gain controller ($G_R = K_R$) results in the following transfer function:

$$\frac{Y(s)}{Y_c(s)} = \frac{G_R G(s)}{1 + G_R G(s)}$$

$$= \frac{\frac{K_R K}{(\tau_1 s + 1)(\tau_2 s - 1)}}{1 + \frac{K_R K}{(\tau_1 s + 1)(\tau_2 s - 1)}}$$

The resulting characteristic equation is :

$$f(s) = (\tau_1 s + 1)(\tau_2 s - 1) + K_R K = 0$$

$$\Rightarrow \tau_1 \tau_2 s^2 + (\tau_2 - \tau_1)s + (K_R K - 1) = 0$$

According to the Routh-Hurwitz criterion, we need all the three terms of the polynomial to be positive. This results in the following conditions:

$$\tau_2 - \tau_1 > 0$$
$$K_R K > 1$$

Thus we can see that the use of a proportional gain controller cannot guarantee the stability of the system as one of the two conditions is entirely dependent on the system properties themselves.

So what happens if we use a PD controller ($G_R = K_R(1 + \tau_D s)$)? The characteristic equation becomes:

$$f(s) = (\tau_1 s + 1)(\tau_2 s - 1) + K_R(1 + \tau_D s)K = 0$$

$$\Rightarrow \tau_1 \tau_2 s^2 + (\tau_2 - \tau_1 + K_R K \tau_D)s + (K_R K - 1) = 0$$

The two Routh-Hurwitz conditions now become:

$$\tau_2 - \tau_1 + K_R K \tau_D > 0$$
$$K_R K - 1 > 0$$

Hence, we can now tweak, both K_R and τ_D to coax the system towards stability.

Exercise 2

The process is described by the following equation:

$$\tau \frac{dy(t)}{dt} + y(t) = Ku(t)$$

In the Laplace domain, this becomes:

$$\tau s Y(s) - \tau y(0) + Y(s) = KU(s)$$

Note that $y(0) = 0$, so the transfer function is given by:

$$G(s) = \frac{Y(s)}{U(s)} = \frac{K}{\tau s + 1}$$

This describes a stable first order system with its pole on the left hand side of the complex plane. Once we add a PI controller to the system ($G_R = K_R(1 + \frac{1}{\tau_I s})$), The resulting closed loop system becomes:

$$\frac{Y(s)}{Y_C(s)} = \frac{G_R G(s)}{1 + G_R G(s)}$$

(a) The stability of the system is determined by the roots of its characteristic equation given by:

$$f(s) = 1 + G_R G(s) = 0$$

$$\Rightarrow 1 + \frac{K_R(\tau_I s + 1)}{\tau_I s} \cdot \frac{K}{\tau s + 1} = 0$$

$$\Rightarrow \tau_I s(\mathrm{\tau s} + 1) + K_R K(\mathrm{\tau_I} s + 1) = 0$$

$$\Rightarrow \tau_I \tau s^2 + \tau_I s + K_R K \tau_I s + K_R K = 0$$

$$\Rightarrow \tau_I \tau s^2 + \tau_I(1 + K_R K)s + K_R K = 0$$

Once again, we can apply the Routh-Hurwitz criteria here. All the three coefficients should be strictly positive. In this case, this leaves us with the following:

$$\tau_I \tau > 0$$

$$\tau_I(1 + K_R K) > 0$$

$$K_R K > 0$$

Since τ_I and τ are time constants, they are already positive. This leaves us with only one criterion for stability: $K_R K > 0$. This essentially means that the controller's mode of action should be dictated by the gain of the system, with $K_R < 0$ if $K < 0$ and vice-versa.

(b) The application of a pure time delay to the system is represented in the Laplace domain by multiplying the original $G(s)$ by $e^{-\theta s}$.

The characteristic equation then becomes

$$f(s) = \ 1 + \frac{K_R(\tau_I s + 1)}{\tau_I s} \cdot \frac{Ke^{-\theta s}}{\tau s + 1} = 0$$

We can approximate the exponential function using a Taylor expansion:

$$e^{-\theta s} = 1 - \theta s + \theta^2 s^2 + \cdots$$

$$\Rightarrow e^{-\theta s} \cong 1 - \theta s$$

Applying this to the characteristic equation, we get:

$$f(s) = \ 1 + \frac{K_R(\tau_I s + 1)}{\tau_I s} \cdot \frac{K(1 - \theta s)}{\tau s + 1} = 0$$

$$\Rightarrow \tau_I s(\tau s + 1) + K_R K(\tau_I s + 1)(1 - \theta s) = 0$$

$$\Rightarrow \tau_I(\tau - \theta K K_R)s^2 + \big(\tau_I + K_R K(\tau_I - \theta)\big)s + K_R K = 0$$

The Routh-Hurwitz conditions for stability are:

$$\tau_I(\tau - \theta K K_R) > 0 \Rightarrow K_R K < \frac{\tau}{\theta}$$

$$\tau_I + K_R K(\tau_I - \theta) > 0 \Rightarrow \tau_I > \frac{K_R K \theta}{1 + K_R K}$$

$$K_R K > 0$$

The first and third conditions together constrain the operating ranges of the gain of the controller. It must have the correct sign depending on the gain of the unregulated system. In addition, it must also take into account the degree of time delay of the original system, with larger time delays necessitating a smaller degree of control (K_R) .

The second condition relates the dynamic response of the integrator part of the controller to the time delay (θ) of the original system, with larger time delays corresponding to a slower, and weaker, effect of the controller.

Exercise 3

The reaction inside the tank is of the form $v_R = kc_B$; since A is in abundance, the reaction rate is only determined by the concentration of B. The mass balances of B and C are given below:

$$\frac{dn_B}{dt} = V \cdot \frac{dc_B}{dt} = qc_{B0} - Vkc_B$$

$$\frac{dn_C}{dt} = V \cdot \frac{dc_C}{dt} = Vkc_B$$

In the Laplace domain, these equations become:

$$sC_B(s) - c_B(0) = \frac{Q(s)}{V} c_{B0} - kC_B(s)$$

$$sC_C(s) - c_C(0) = kC_B(s)$$

The question says that A is in excess and B is added slowly. Hence we can assume that $c_B(0) = 0 = c_C(0)$.

We then have

$$V(s+k)C_B(s) = Q(s)c_{B0} \;\; \Rightarrow \;\; \frac{C_B(s)}{Q(s)} = \frac{c_{B0}}{V(s+k)}$$

$$sC_C(s) = kC_B(s) \;\; \Rightarrow \;\; \frac{C_C(s)}{C_B(s)} = \frac{k}{s}$$

$$\Rightarrow \frac{C_C(s)}{Q(s)} = \frac{C_C(s)}{C_B(s)} \cdot \frac{C_B(s)}{Q(s)}$$

$$= \frac{k \cdot c_{B0}}{V(s+k)s}$$

The above transfer function is an integrator with a pole at 0, and as such is an unstable system. In such a system, a bounded input, such as a step increase in the incoming flow, q, will result in an indefinite increase in the concentration of C.

If you want to see this clearly, we can decompose the transfer function into:

$$G(s) = \frac{K}{(s+k)s} = \frac{A}{s} + \frac{B}{s+k}$$

The output to any input $q(t)$ is given by the inverse Laplace transform of :

$$C_c(s) = G(s) \cdot Q(s) = A \cdot \frac{Q(s)}{s} + B \cdot \frac{Q(s)}{s+k}$$

The first of these two terms is the Laplace transform of an integration function!